(Anti-)self-dual homogeneous vacuum gluon field as an origin of confinement and $\mathrm{SU_L(N_F)} \times \mathrm{SU_R(N_F)}$ symmetry breaking in QCD

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Abstract. It is shown that an (anti-)self-dual homogeneous vacuum gluon field appears in a natural way within the problem of calculation of the QCD partition function in the form of Euclidean functional integral with periodic boundary conditions. There is no violation of cluster property within this formulation, nor are parity, color and rotational symmetries broken explicitly. The massless limit of the product of the quark masses and condensates, $m_f\langle\bar{\psi}_f\psi_f\rangle$, is calculated to all loop orders. This quantity does not vanish and is proportional to the gluon condensate appearing due to the nonzero strength of the vacuum gluon field. We conclude that the gluon condensate can be considered as an order parameter both for confinement and chiral symmetry breaking.

I Introduction

The purpose of this paper is to find a regular procedure for construction of the Euclidean QCD generating functional under assumption that the homogeneous (anti-)self-dual gluon field realizes an absolute minimum of the QCD effective action, and, on the basis of this procedure, to discuss a relation of the homogeneous (anti-)self-dual gluon field to the issues of confinement and chiral symmetry breaking in QCD.

The physical picture of nonperturbative QCD vacuum realised with the (anti-)self-dual homogeneous gluon field has become prominent since the early eighties, when Leutwyler demonstrated the stability of this gluon configuration against local quantum fluctuations and noticed, that this field could be related to the problems of confinement and chiral symmetry breaking [1,2]. Elizalde and Soto, and many other authors have obtained strong evidence that this field could be a true minimum of the QCD effective potential (see [3,4] and references therein). Manifestations of this gluon configuration in the spectrum and weak decays of light mesons, their excited states, heavy quarkonia and heavy-light mesons were studied in recent papers [5,6]. The vacuum field under consideration produces several qualitative regimes for masses and decay constants which are completely consistent with experimental data. Namely, the masses of light pseudoscalar and vector mesons are strongly split, orbital and radial excitations of light mesons show Regge behaviour, the mass of heavy quarkonium tends to be equal to sum of the masses of quarks, the heavy-light meson mass approaches the mass of the heavy quarks, and the weak decay constant for pseudoscalar heavy-light mesons has asymptotic behaviour $1/\sqrt{m_Q}$. Moreover, scalar and axial mesons are absent in the spectrum as simple $q\bar{q}$ states, but appear in the super-fine structure of orbital excitations of vector mesons. Quantitatively, the masses and decay constants of mesons from all different regions of the spectrum are described within ten percent inaccuracy. These different phenomena are displayed with the minimal set of parameters: gauge coupling constant, strength of the vacuum field and the quark masses.

This short review underlines that a series of results concerning both the QCD ground state and the meson phenomenology does support the field under consideration as a possible agent for confinement and chiral (flavour) symmetry breaking. However, there are three essential gaps that hinder justification of this physical picture. Regular formulation of the problem about QCD ground state, realized by the (anti) -self-dual homogeneous field, is missed. There is no proof that this field minimizes the QCD effective potential. Most of the results concerning the field under consideration are obtained within the one-loop approximation. In this paper, we attempt to fill in the first and third gaps.

We show that the analogy between Euclidean functional integrals and the partition function of statistical systems in the thermodynamic limit leads to the Euclidean QCD generating functional including the vacuum field under consideration in the form, that does not show an explicit violation of the space-time and color symmetries and

ensures the cluster decomposition property of the correlators of quantum fields.

Using this representation, we investigate the massless limit of the renormalization group invariant quantity $m\langle\bar{\psi}\psi\rangle_\mathcal{B}$ which is a product of the quark mass and quark condensate in the presence of vacuum (anti-)self-dual homogeneous gluon field. An important result of this work is the formula

$$\sum_{f=1}^{N_F} \lim_{m_f\to 0} m_f\langle\bar{\psi}_f\psi_f\rangle_\mathcal{B} = -N_F\frac{\mathcal{B}^2}{2\pi^2}, \qquad (1)$$

where $\mathcal{B}$ is the strength of the vacuum field (gauge coupling constant is included into $\mathcal{B}$). Equation (1) is valid to all loop orders. Zero modes of Dirac operator in the presence of the (anti-)self-dual gluon field and the homogeneity of this field are the key reasons for relation (1). This equation shows a non-Goldstone mechanism of $SU_L(N_F)\times SU_R(N_F)$ symmetry breaking, which appears as a secondary effect of spontaneous violation of parity. At the same time, since the parity is not broken explicitly, the $U_A(1)$ symmetry is preserved by the vacuum field under consideration. Furthermore, the homogeneity and (anti-)self-duality of the field leads to the confinement of quarks and most of gluon degrees of freedom, which can be seen as the absence of poles in propagators (in the momentum representation). Moreover, a homogeneity of the vacuum field means, that at zero and nonzero field we deal with two different phases (in the sense of statistical mechanics) and the gluon condensate plays the role of order parameter in the similar fashion as the scalar field condensate does in the Higgs model.

It is notable that the present paper is closely motivated by our recent studies [6] of the meson properties induced by the vacuum field under consideration. Chiral symmetry breaking due to the zero modes appears to be a dominating factor in forming the masses and decay constants of light mesons, while confinement (the propagators are entire functions) is responsible for Regge trajectories, heavy quarkonia and heavy-light mesons.

II Generating functional

First of all, we need to explain what is hidden behind the symbol $\langle\ldots\rangle_\mathcal{B}$. In other words, what is the formal statement of the problem about QCD ground state, within which this field appears in a natural and self-consistent way?

The usual statement of the problem about vacuum (phase) structure of quantum field systems is based on the analogy between the functional integrals in Euclidean QFT and the partition function of quantum statistical systems in the infinite volume (thermodynamic) limit. Therefore, we need an appropriate representation for the QCD partition function in the infinite volume limit. The most subtle point is a choice of boundary conditions for the functional space of integration. The standard way is to introduce a large space-time box, to impose periodic bound-

ary conditions on the fields in the box and then to study the infinite volume limit. We follow just this prescription.

It should be noted, that condition for the fields to vanish at infinity, which is normal for QFT in the perturbative regime, is not appropriate, since the translation invariant fields are excluded ad hoc. There is no chance to get insight into the critical phenomena of long range correlations. The instanton-like formulation of the problem – to calculate a transition amplitude from the field configuration A given at Euclidean time τ_1 to another configuration A' at time τ_2 [2] – is not suitable either. In this case, the homogeneous field comes through the boundary conditions, which results in a hard violation of the cluster property and explicit breakdown of rotational and color symmetries and parity.

Let us start with pure gluodynamics. A naive representation for partition function looks like

$$Z \sim \int_{\mathcal{F}_{L,\beta}} D\mathcal{A}\exp\left\{\int_V d^4x\mathcal{L}_{\text{YM}}(\mathcal{A})\right\}, \qquad (2)$$

where V is a large Euclidean volume, L and $\beta^{-1} = T$ are the space box size and the temperature. The functional space $\mathcal{F}_{L,\beta}$ contains gauge fields $\mathcal{A}_\mu$ satisfying periodic boundary condition. Notice, that translation invariant fields, in particular the (anti-)self-dual homogeneous field

$$B_\mu(x) = \frac{1}{2}nB_{\mu\nu}x_\nu, \quad n = \lambda_3\sin\zeta + \lambda_8\cos\zeta, \qquad (3)$$

$$\tilde{B}_{\mu\nu} = \pm B_{\mu\nu}, \quad B_{\mu\rho}B_{\rho\nu} = -\delta_{\mu\nu}B^2, \quad B^2 = \text{const},$$

belong to $\mathcal{F}_{L\beta}$. In case (3), an arbitrary translation

$$B_\mu(x+\xi) = B_\mu(x) + \partial_\mu\omega(x,\xi), \quad \omega = x_\nu B_\nu(\xi), \qquad (4)$$

can be compensated by a suitable gauge transformation.

Field configuration (3) is not a dynamical variable in the sense, that its equation of motion does not contain any derivatives, but is just a constraint. This field must be integrated out if one looks for an integral representation for partition function which corresponds to an actual ground state of the system. However, this integration should be based on resolving the constraint which takes into account all quantum corrections coming from the dynamical modes of the gauge fields. The quantum constraint can have solutions that are neither visible at the classical level nor within the perturbation theory. At the same time, these nontrivial solutions for the constant fields (condensates) govern critical phenomena in systems with the infinite number of degrees of freedom. One can easily illustrate this statement by the phase transitions in the models with ϕ^4 and Yukawa interactions (e.g., see [7] and references therein).

The integral over the homogeneous field can be separated in (2) with a simultaneous fixing of a gauge of dynamical fields by means of the Faddev-Popov trick:

$$1 = \int_0^\infty dB \int_{\Sigma_B} d\sigma_B\ \Phi[\mathcal{A},B] \int_\mathcal{F} DA \int D\omega$$
$$\times\delta\left[\mathcal{A}^\omega - A^\omega - B^\omega\right]\delta\left[\nabla(B^\omega)A^\omega\right], \qquad (5)$$

where the space $\bar{\mathcal{F}}$ does not contain the nondynamical mode (3), B is the homogeneous field strength, $\nabla_\mu(B) = \partial_\mu - iB_\mu(x)$ denotes a covariant derivative in the adjoint representation. The coupling constant g is included into the field B_μ.

The measure $d\sigma_B$ is defined as

$$\int_{\Sigma_B} d\sigma_B = \frac{1}{(4\pi)^2} \sum_{\pm} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\zeta = 1. \quad (6)$$

Definition (6) corresponds to integration over the spatial (spherical) angles (φ, θ) of the field (3) and angle ζ which defines its orientation in color space (in the diagonal representation of matrix n in (3)). The sign '$\pm$' corresponds to summation of the self- and anti-self-dual configurations. The final representation for Z is then

$$Z = \lim_{\Lambda \to \infty} \mathrm{R}_\Lambda N \int_{\Sigma_B} d\sigma_B \int_0^\infty dB \int_{\mathcal{F}} DA$$
$$\times \Delta_{\mathrm{FP}}[B, A] \delta\left[\nabla(B)A\right] \exp\left\{\int_V d^4x \mathcal{L}_{\mathrm{YM}}(A + B)\right\}, \quad (7)$$

where $\Delta_{\mathrm{FP}}[B, A]$ is the Faddeev-Popov determinant for the background gauge condition $\nabla(B)A = 0$. An appropriate regularization R_Λ of ultraviolet divergences and renormalization prescription are implied in (7). By definition, an integral over the fields A gives rise to an effective potential of the field B_μ. The background field does not affect general renormalizability of the theory [8,9], and we rewrite Z in the form

$$Z = N' \int_{\Sigma_B} d\sigma_B \int_0^\infty dB \exp\left\{-VU_{\mathrm{eff}}[B^2; g(\mu), \mu, \beta]\right\}, \quad (8)$$

where μ is the renormalization point. As has been mentioned, the background field in (8) includes the coupling constant $B \equiv gB$. In the background gauge, the composition gB is RG-invariant [9]. Furthermore, the effective potential is invariant under gauge and parity transformations, and space rotations (this also follows from the general background field method). Thus, we arrive at the expression

$$Z = N' \int_0^\infty dB \exp\left\{-VU_{\mathrm{eff}}[B^2; g(\mu), \mu, \beta]\right\}.$$

If the effective potential has a minimum at nonzero field strength $\mathcal{B} = \mathcal{B}(g(\mu), \mu, \beta)$, then, in the infinite volume limit, the saddle-point method gives

$$Z = \exp\left\{-VF[g(\mu), \mu, \beta]\right\},$$
$$F = U_{\mathrm{eff}}[\mathcal{B}^2(g(\mu), \mu, \beta); g(\mu), \mu, \beta] < 0.$$

The free energy density F is RG-invariant. For zero temperature, $\mathcal{B}$ is nothing other than the RG-invariant combination of the running coupling constant $g(\mu)$ and the renormalization point μ, hence:

$$\lim_{\beta \to \infty} \mathcal{B}^2 = C_{\mathcal{B}} \, \Lambda_{\mathrm{QCD}}^4, \quad \lim_{\beta \to \infty} F = -C_F \Lambda_{\mathrm{QCD}}^4, \quad (9)$$

where $C_{\mathcal{B}}$ and C_F are positive numbers, and

$$\Lambda_{\mathrm{QCD}}^2 = \mu^2 \exp\left\{\int^{g(\mu)} \frac{dg}{\beta(g)}\right\}.$$

These equations link the strength of the vacuum field with the "fundamental scale" Λ_{QCD} (see also [2,4,10]).

Now we can represent the partition function Z in the form of functional integral over the quantum fluctuation fields A, that does not contain the translation invariant mode:

$$Z = \lim_{\Lambda \to \infty} \mathrm{R}_\Lambda N \int_{\Sigma_{\mathcal{B}}} d\sigma_{\mathcal{B}} \int_{\bar{\mathcal{F}}} DA \, \Delta_{\mathrm{FP}}[\mathcal{B}, A] \delta\left[\nabla(\mathcal{B})A\right]$$
$$\times \exp\left\{\int_V d^4x \mathcal{L}_{\mathrm{YM}}(A + \mathcal{B})\right\}. \quad (10)$$

Equation (10) gives the representation that we are looking for. It is based on the strong but single assumption that the (anti-)self-dual field corresponds to the global minimum of the QCD effective action. Lattice calculation of the effective potential for different translation invariant gluon fields seems to be the most direct way to verify this assumption. The study of the effective potential for the pure chromomagnetic homogeneous field in the three-dimensional $SU(2)$ lattice gauge theory gives an example of the lattice approach to the problem. The first evidence from lattice simulations was obtained of the existance of a nontrivial minimum in the effective potential [11]. However, the (anti-)self-dual field in $(3 + 1)$ dimensions is a particularly interesting configuration due to other reasons to be discussed below (see also [1,2]).

On the basis of representation (10), the QCD generating functional for correlation functions in the infinite volume and at zero temperature has to be defined as

$$Z_{\mathcal{B}}[J, \eta, \bar{\eta}] =$$
$$\lim_{\Lambda \to \infty} \mathrm{R}_\Lambda \mathcal{N}_{\mathcal{B}} \int_{\Sigma_{\mathcal{B}}} d\sigma_{\mathcal{B}} \int_{\bar{\mathcal{F}}} D\mu_A(A, \mathcal{B}) \int_{\mathcal{G}} \prod_f D\psi_f D\bar{\psi}_f$$
$$\exp\left\{\int d^4x \bar{\psi}_f(x) \left[i\hat{\nabla} - m_f + g\hat{A}\right] \psi_f(x)\right.$$
$$\left. + i \int d^4x \left[JA + \bar{\eta}_f\psi_f + \bar{\psi}_f\eta_f\right]\right\},$$
$$D\mu_A(A, \mathcal{B}) = DA\Delta_{\mathrm{FP}}[\mathcal{B}, A]\delta\left[\nabla(\mathcal{B})A\right] \quad (11)$$
$$\times \exp\left\{\int d^4x \mathcal{L}_{\mathrm{YM}}[A + \mathcal{B}]\right\},$$
$$\hat{A} = \gamma_\mu A_\mu, \quad \hat{\nabla} = \gamma_\mu \nabla_\mu, \quad \nabla_\mu = \partial_\mu - i\mathcal{B}_\mu.$$

The constant $\mathcal{N}_{\mathcal{B}}$ provides the standard normalization $Z_{\mathcal{B}}[0, 0, 0] = 1$. The functional space $\bar{\mathcal{F}}$ contains the gauge

fields vanishing at infinity. The change of boundary conditions (vanishing fields instead of periodic ones) is unimportant for physics, since quantum fluctuations A does not contain translation invariant modes. We have also added massive quarks. The fermionic functional integral spans the Grassmann algebra $\mathcal{G}$ of square integrable fields. To be more precise, we will define this integral via a decomposition of the fields $\bar{\psi}$ and ψ over the eigenmodes ψ_n of the Dirac operator in the presence of vacuum gluon field $\mathcal{B}$ (an anti-hermitian representation for the γ-matrices in Euclidean space is used)

$$-i\hat{\nabla}\psi_n(x) = i\lambda_n\psi_n(x). \qquad (12)$$

As a matter of fact, this definition of the fermionic integral implies, that the ground state of the system is governed by the vacuum field $\mathcal{B}$, and the interaction $\bar{\psi}\hat{A}\psi$ of quarks with the quantum gauge field A has to be treated as perturbation. Now, let us seek insight into the properties of representation (11).

III Parametrization, cluster property, symmetries

The generating functional (11) contains the intrinsic dimensionful quantity $\mathcal{B}$ (see also (9)), which provides the natural reference scale for running quark masses $\bar{m}_f(\mu)$ and gauge coupling constant $\bar{\alpha}_s(\mu)$. Therefore, the strength of the vacuum field $\mathcal{B}$, the quark masses and coupling constant at the scale $\mu = \sqrt{\mathcal{B}}$ can be considered as the physical (intrinsic) parameters of QCD in the representation (11). Values of the parameters can be extracted from the analysis of hadron spectrum (e.g., see [6]).

Correlation functions for the local or nonlocal operators $\mathcal{O}_j[A, \psi, \bar{\psi}]$ composed of the quantum fields are given by the standard formula

$$\langle \mathcal{O}_1[A, \psi, \bar{\psi}] \ldots \mathcal{O}_n[A, \psi, \bar{\psi}]\rangle_{\mathcal{B}} = \left(\mathcal{O}_1\left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta\eta}\right] \cdots \right.$$

$$\left. \ldots \mathcal{O}_n\left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta\eta}\right] Z_{\mathcal{B}}[J, \eta, \bar{\eta}]\right)_{J=\eta=\bar{\eta}=0}. \qquad (13)$$

It should be stressed that $A \in \bar{\mathcal{F}}$, and the functional space $\bar{\mathcal{F}}$ does not contain the homogeneous fields. The gauge field A in (13) and (11), being a quantum fluctuation under the background field, vanishes at the spacetime infinity. The connected Green's functions, given by the derivatives of $W_{\mathcal{B}} = -\ln Z_{\mathcal{B}}[J, \eta, \bar{\eta}]$ go to zero at the infinity as well. Thus, the correlators are defined by (13) as the averages of the products of quantum field fluctuations over the physical vacuum state, and ensure the cluster decomposition property. In particular, the correlator of the pseudoscalar winding number density at the points x and y tends to zero in the limit $|x - y| \to \infty$:

$$\langle \phi(x)\phi(y)\rangle_{\mathcal{B}} = \langle\phi(x)\rangle_{\mathcal{B}}\langle\phi(y)\rangle_{\mathcal{B}} \equiv 0,$$
$$\phi(x) = \mathrm{Tr}F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x), \qquad (14)$$
$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + ig[A_\mu, A_\nu].$$

Due to the integration over the angular variables $\Sigma_{\mathcal{B}}$ and summation of the self- and anti-self-dual configurations, the parity, rotational and color symmetries are not broken explicitly. The correlators depend on the strength $\mathcal{B}$ of the vacuum field but not on its direction, which is random and not observable. They have the same tensor structure as in the absence of the vacuum field. At the same time, violation of the symmetries is seen in the integrand of (11), which is an indication of spontaneous breaking of the above-mentioned symmetries. Thus, we meet an unusual mechanism of SSB due to the condensation of the vector bosons.

The order parameter for the phase with a nonzero vacuum homogeneous field is obvious. This is the lowest non-vanishing gluon condensate, defined as

$$\langle\left[\partial_\nu\mathcal{A}_\mu^a(x) - \partial_\mu\mathcal{A}_\nu^a(x)\right]^2\rangle = 4\mathcal{B}^2 + (\text{pert.corr.}). \quad (15)$$

Here $\mathcal{A} = (A + B) \in \mathcal{F}$, and $\langle\ldots\rangle$ denotes an averaging by means of Eq. (7). However, any of the correlators

$$\langle\mathcal{O}_1[\mathcal{A}] \cdot \ldots \cdot \mathcal{O}_n[\mathcal{A}]\rangle$$

$$= \int_{\Sigma_{\mathcal{B}}} d\sigma_{\mathcal{B}}\mathcal{O}_1[\mathcal{B}] \cdot \ldots \cdot \mathcal{O}_n[\mathcal{B}] + \ldots, \qquad (16)$$

which contains a constant part, can be taken as order parameter.

It should be stressed, that the cluster decomposition is a property of correlators of quantum fluctuations. The correlators of quantum fluctuations under the background field defined by (13) and (11) ensure this property. The correlators (15) and (16) are defined in a different way. The field $\mathcal{A}$ is a sum of the vacuum field B and the quantum fluctuation field A ($\mathcal{A} = B + A$). They contain a constant part and act as order parameters concerning the long-range correlations in the QCD vacuum. This is in a complete analogy with the scalar field condensate in the Higgs model.

IV Chiral symmetries

Spontaneous violation of parity should influence the chiral symmetries $U_A(1)$ and $SU_L(N_F) \times SU_R(N_F)$. Due to summing the self- and anti-self-dual configurations in (11), the vacuum expectation value of a pseudo-tensor operator is identically equal to zero. In particular, $U_A(1)$ symmetry is not broken in the sense that

$$\langle\partial_\mu\bar{\psi}_f(x)\gamma_5\gamma_\mu\psi_f(x)\rangle_{\mathcal{B}} \equiv 0.$$

An explicit violation of parity as in the instanton θ-vacuum is needed.

To study the flavour chiral symmetry $SU_L(N_F) \times SU_R(N_F)$ let us consider the massless limit of composition of the quark masses and quark condensates:

$$\sum_{f=1}^{N_F} m_f\langle\bar{\psi}_f(x)\psi_f(x)\rangle_{\mathcal{B}}$$

$$= \sum_{f=1}^{N_F} m_f\left[\frac{\delta}{i\delta\eta_f(x)}\frac{\delta}{i\delta\bar{\eta}_f(x)}Z_{\mathcal{B}}[\eta, \bar{\eta}, J]\right]_{\bar{\eta}=\eta=J=0}. \qquad (17)$$

The nontrivial point in the calculation of this quantity consists in the following. In the massless limit $m_f \to 0$, the divergent contributions $O(m_f^{-k})$, with k being some positive integer, can appear potentially at any loop order. This means, that the perturbation decomposition can fail in the massless limit. In this case, the divergent terms have to be summed to all loop orders. This infrared problem comes from the zero modes ψ_0 of (12) with $\lambda_0 = 0$ existing due to an (anti-) self-duality of the vacuum field. It should be noted, that this problem arises both for the homogeneous and instanton fields. However, the crucial difference between the homogeneous field and the instanton θ-vacuum consists in the normalization of the generating functional. Unlike the instanton case (e.g., see [12,13]), the normalization constant $\mathcal{N}_\mathcal{B}$ in (11) corresponds to the nonperturbative vacuum and contains the contribution of the fermion zero modes. Therefore, in the massless limit, no problem arises with the fermion determinant coming from the integral in (11); it is cancelled by the normalization constant $\mathcal{N}_\mathcal{B}$.

Now we will show that a singularity $1/m$ exists in the lowest one-loop diagram for the quark condensate but does not appear in higher orders. This allows one to calculate the massless limit of (17) explicitly and to prove relation (1). The most direct way consists in the following.

First of all, notice that (17) can be rewritten in the equivalent form

$$\sum_{f=1}^{N_F} m_f \langle \bar{\psi}_f(x)\psi_f(x) \rangle_\mathcal{B}$$

$$= -\mathcal{Z}_\mathcal{B}^{-1}(m) \sum_{f=1}^{N_F} \lim_{V \to \infty} V^{-1} m_f \frac{d}{dm_f} \mathcal{Z}_\mathcal{B}(m), \qquad (18)$$

$$\mathcal{Z}_\mathcal{B}(m) =$$

$$\lim_{\Lambda \to \infty} \mathrm{R}_\Lambda \mathcal{N}_\mathcal{B}(\mu) \int_{\Sigma_\mathcal{B}} d\sigma_\mathcal{B} \int_{\bar{\mathcal{F}}} D\mu_A(A,\mathcal{B}) \int_\mathcal{G} \prod_f D\psi_f D\bar{\psi}_f$$

$$\exp\left\{ \int d^4x \bar{\psi}_f(x) \left[i\hat{\nabla} - m_f + g\hat{A} \right] \psi_f(x) \right\}, \qquad (19)$$

where the normalization constant $\mathcal{N}_\mathcal{B}(\mu)$ is taken so that $\mathcal{Z}_\mathcal{B}(\mu) = 1$ for some $\mu \neq 0$. The LHS of (18) does not depend on $\mathcal{N}_\mathcal{B}(\mu)$, but this normalization provides us with an appropriately defined integral under the derivative.

Consider for a moment the one-flavour case. An extension to $N_F > 1$ is straightforward. Formal integration over the quark field in the partition function gives

$$Z_\mathcal{B}(m) = \lim_{\Lambda \to \infty} \mathrm{R}_\Lambda \mathcal{N}_\mathcal{B}(\mu) \int_{\Sigma_\mathcal{B}} d\sigma_\mathcal{B} \int_{\bar{\mathcal{F}}} D\mu_A(A,\mathcal{B})$$

$$\times \det\left[-i\hat{\nabla} + m - g\hat{A} \right]. \qquad (20)$$

Our definition of the fermion integral via the eigenmodes of Dirac operator means the determinant in (20) and its derivative in (18) are defined as

$$\det\left[-i\hat{\nabla} + m - g\hat{A} \right] = \det\left[-i\hat{\nabla} + m \right] \det\left[1 - g\hat{A}S \right], \qquad (21)$$

$$\frac{d}{dm} \det\left[-i\hat{\nabla} + m - g\hat{A} \right]$$

$$= \det\left[-i\hat{\nabla} + m \right] \det\left[1 - g\hat{A}S \right]$$

$$\times \left[\tilde{\mathrm{Tr}}S + \frac{d}{dm} \tilde{\mathrm{Tr}}\ln(1 - g\hat{A}S) \right], \qquad (22)$$

where the trace $\tilde{\mathrm{Tr}}$ includes the space-time integration, and the quark propagator $S(x,y)$ satisfies the equation

$$\left(i\hat{\nabla}_x - m \right) S(x,y) = -\delta(x-y). \qquad (23)$$

The term $\tilde{\mathrm{Tr}}S$ in (22) is the lowest order contribution to the quark condensate. Higher perturbation corrections come from the quark loops contained in the logarithmic term in (22). The decisive point is that these quark loops are regular in the massless limit, while the lowest term is singular:

$$\lim_{m \to 0} \tilde{\mathrm{Tr}}\ln(1 - g\hat{A}S) \sim 1 + O(m),$$

$$\lim_{m \to 0} \tilde{\mathrm{Tr}}S \sim \frac{1}{m} + O(1). \qquad (24)$$

It is notable, that, in another context, a regularity of the simplest two-gluon loop was demonstrated by Flory [14].

Using the standard representation for Green's function in terms of the matrix elements of the projection operators $\mathcal{P}_n$

$$S(x,y) = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n(x,y)}{m + i\lambda_n},$$

one can separate the contribution of the zero eigenmodes and normal modes to the propagator

$$S(x,y) = S'(x,y) + S_0(x,y),$$

$$S'(x,y) = i\overrightarrow{\hat{\nabla}}_x \Delta(x,y)P_\pm + \Delta(x,y) i\overleftarrow{\hat{\nabla}}_y P_\mp$$

$$+ O(m), \qquad (25)$$

$$\overrightarrow{\nabla} = \overrightarrow{\partial} - i\mathcal{B}, \qquad \overleftarrow{\nabla} = \overleftarrow{\partial} + i\mathcal{B},$$

$$S_0(x,y) = \mathcal{P}_0(x,y)/m. \qquad (26)$$

Representation (25) for the normal mode propagator was obtained by Brown et al. [15] for an arbitrary (anti-)self-dual background field (see also [16] and Appendix). Here $\Delta(x,y) = f(x,y)/4\pi^2(x-y)^2$ is the scalar massless propagator in the background field (3), n is a diagonal matrix (see (3)), and $P_\pm = (1 \pm \gamma_5)/2$. The projector onto the zero mode subspace

$$\mathcal{P}_0 = \frac{n^2\mathcal{B}^2}{4\pi^2} f(x,y) P_\mp \Sigma_\mp, \qquad (27)$$

$$\Sigma_\pm = \frac{1}{2}\left(1 \pm \Sigma_j b_j\right), \qquad \Sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_{jk},$$

$$\sigma_{jk} = [\gamma_j, \gamma_k]/2i, \qquad (28)$$

$$b_j = B_j/B, \ B_i = -\frac{1}{2}\varepsilon_{ijk}B_{jk}, \ i,j,k = 1,2,3$$

$$f(x,y) = \exp\left\{-\frac{1}{4}\sqrt{n^2}\mathcal{B}(x-y)^2 + \frac{i}{2}nx_\mu\mathcal{B}_{\mu\nu}y_\nu\right\}, \tag{29}$$

$$\int d^4z \mathcal{P}_0(x,z)\mathcal{P}_0(z,y) = \mathcal{P}_0(x,y),$$

is defined by the square integrable solution

$$\psi_0(x,x_0) = \frac{n^2\mathcal{B}^2}{4\pi^2}uf(x,x_0) \tag{30}$$

of the equation (12) with $\lambda_0 = 0$. Details of calculation of ψ_0 can be found in Appendix. The spinor u in (30) is an eigenvector of the matrices $\Sigma_j b_j$ and γ_5:

$$\Sigma_j b_j \psi_0 = \mp\psi_0, \ \gamma_5\psi_0 = \mp\psi_0,$$

which is the well-known [12,15,16] property of zero modes to be left-handed in a self-dual field and right-handed in an anti-self-dual field. As a result, the projector

$$\mathcal{P}_0(x,y) = \int d^4x_0 \psi_0(x,x_0)\psi_0^\dagger(y,x_0) \tag{31}$$

contains the projection matrices P_- (self-dual field) or P_+ (anti-self-dual field) and Σ_- ($n > 0$) or Σ_+ ($n < 0$). The matrix Σ_+ (Σ_-) can be seen as the projector onto the quark state with the spin orientated along (against) the chromomagnetic field B_j.

The space-time point x_0 describes a position of the fermion "pseudoparticle" $\psi_0(x,x_0)$. We see that the zero eigenvalue is infinitely degenerate which is a manifestation of the above-mentioned invariance of the vacuum field under translations and simultaneous gauge transformations (see Appendix). This feature, as well as the functional form of fermion zero mode (29), (30), is very similar to the properties of Leutwyler's chromons [1,2] which are gluon zero modes in the same background field.

Now we can return to (22) and represent the logarithmic term in the form

$$\tilde{\mathrm{Tr}}\ln(1 - g\hat{A}S) = -\sum_{k=1}^{\infty}\frac{g^k}{k}\tilde{\mathrm{Tr}}\left[\hat{A}(S_0 + S')\right]^k. \tag{32}$$

Then one makes use of (26) and (27) to notice that

$$S_0(x,y)\gamma_\mu A_\mu(y)S_0(y,z) \equiv 0 \tag{33}$$

due to the projectors $P_\mp$ in S_0 and commutativity of the matrix $P_\mp$ with Σ_- and Σ_+. Therefore, all the terms in (32), which contain the block $S_0\gamma_\mu A_\mu S_0$, vanish. Furthermore, any two propagators $S_0 \sim P_\mp$ in the rest of terms of (32) are separated by an odd number of vertices γA and propagators $S' \sim (\gamma + O(m))$. Hence, the terms in (32) with nonzero trace of γ-matrices contain at least one quark mass m in the numerator for each m in the denominator. In other words, there is always an odd number of γ-matrices between two chiral projectors $P_\mp$ in the singular terms, and the quark loops are finite in the limit $m \to 0$, as is pointed out in (24).

Finally, taking into account equations (18), (20), (22) and (26)-(29), we arrive at

$$\lim_{m\to 0} m\langle\bar{\psi}\psi\rangle_\mathcal{B} = -\lim_{m\to 0}mV^{-1}\tilde{\mathrm{Tr}}S$$

$$= -V^{-1}\int_V d^4x \mathrm{Tr}\mathcal{P}_0(x,x) = -\frac{\mathcal{B}^2}{2\pi^2} \tag{34}$$

for one flavour. For several flavours one gets formula (1). Thus, the gluon condensate (15) can be considered as an order parameter for the flavour chiral symmetry breaking.

The nonzero massless limit of $m\langle\bar{\psi}\psi\rangle_\mathcal{B}$ indicates a non-Goldstone mechanism of symmetry breaking. From our point of view, breakdown of the chiral symmetry appears here as a secondary effect of spontaneous violation of parity, which is a discrete symmetry. Since zero modes (30) are right-handed in the anti-self-dual field and left-handed in the self-dual field, hence in both terms ($\pm$) of generating functional (11) the chiral group is reduced to one of the flavour subgroups

$$SU_L(N_F) \times SU_R(N_F) \longrightarrow SU_L(N_F) \ (\text{or } SU_R(N_F))$$

for the zero mode component of the fermion fields

$$\chi_0(x) = \int d^4z q_0(z)\psi_0(x,z), \ \bar{\chi}_0(x) = \int d^4z\bar{q}_0(z)\psi_0^\dagger(x,z),$$

where $(q_0, \bar{q}_0)$ are the basic elements of the zero mode subspace of the Grassmann algebra $\mathcal{G}$ in (11). As has been mentioned, due to the translation invariance of the vacuum field, there is a continuum of fermion zero modes, and their condensation in the infinite volume produces a very strong effect. Consequences of this effect in meson phenomenology are discussed in [6]. In particular, a contribution of zero modes to the quark propagator splits the masses of pseudoscalar (π, K) and vector (ρ, K^*) mesons and leads to correct values of weak decay constants of the pseudoscalar mesons.

V Confinement

Now we will comment briefly on the quark confinement produced by the field under consideration. The quark propagator S being the solution to (23) (see Appendix and [5, 6,17]) can be written in the form

$$S(x,y) = e^{\frac{i}{2}nx_\mu B_{\mu\nu}y_\nu}H(x-y),$$

$$H(z) = \left(m + i\hat{\nabla}\right)\frac{B\sqrt{n^2}}{8\pi^2}\int_0^1\frac{dt}{(1-t)^2}$$

$$\times \exp\left\{-\sqrt{n^2}Bz^2(1+t)/4(1-t)\right\}$$

$$\times \left[P_\pm t^{m^2/2B\sqrt{n^2}} + P_\mp\Sigma_+ t^{m^2/2B\sqrt{n^2}+n/\sqrt{n^2}}\right.$$

$$\left. +P_\mp\Sigma_- t^{m^2/2B\sqrt{n^2}-n/\sqrt{n^2}}\right]. \tag{35}$$

Fourier transform of the translation invariant part H is an entire analytical function in the complex momentum

plane. This means that there are no poles corresponding to free quarks. The other side of this peculiarity is that the Dirac equation for massive quarks in the presence of the background field (3)

$$\left(i\hat{\nabla}_x - m_f\right)\psi(x) = 0$$

has only the trivial solution $\psi \equiv 0$. Therefore, one has no appropriate field to construct asymptotic free states for quarks. In this sense, the quarks cannot exist as free particles but can propagate as virtual objects. The characteristic scale of propagation of these quark "virtons" is determined by the strength $\mathcal{B}$ of the vacuum gluon field. This situation can be seen as the quark confinement, for which gluon condensate $\mathcal{B}^2$ plays the role of an order parameter. Meantime, nothing preserves these virtons to form a colorless composite particle by means of gluon exchange [5,6]. A colorless bound state does not feel the confining vacuum field and can be observable. A mathematically consistent treatment of this physical concept, especially a reasonable solution of the bound state problem in terms of composite fields, requires an application of the methods of nonlocal quantum field theory [5,6,17,18]. As is shown in [5,6], above picture of confinement provides Regge spectrum of orbital meson excitations.

VI Discussion

In conclusion we would like to mention the "flaws" in this picture. The problem about the minimum of the effective potential is not solved. The lattice calculation of the effective potential for different translation invariant gluon configurations could play the decisive role. To our knowledge, nobody has performed lattice calculations for QCD or pure Yang-Mills theory in the presence of (anti-)self-dual homogeneous field. There are only few calculations for the pure chromomagnetic field ([11] and references theirin). Besides confined modes of the gluon field (in the same sense as for the quarks), free gluons appear to be allowed. At first sight, the gluons, longitudinal in the color space with respect to the vacuum field, seem to be not confined [2,5]. Solution of the $U_A(1)$ problem, which is missed in our consideration, can come from the investigation of the local instanton-like (anti-)self-dual deformations of the homogeneous background field - chromons [2,10]. In the meantime, these "flaws" are problems for further consideration.

Appendix A: Zero mode

Here we will find the explicit form of the solution ψ_0 to (12) corresponding to the zero eigenvalue. Below we use the following notation (also, see (3))

$$\nabla_\mu(x) = \partial - iB_\mu(x), \quad \nabla_\mu^*(x) = \partial + iB_\mu(x)$$

$$B_\mu(x) = \frac{1}{2}nB_{\mu\nu}x_\nu, \quad B_{ij} = -\varepsilon_{ijk}B_k,$$

$$B_{j4} = \pm B_j, \quad b_j = B_j/B,$$

$$\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu], \quad \Sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_{jk}, \quad \Sigma_\pm = \frac{1}{2}\left(1 \pm \Sigma_j b_j\right),$$

$$P_\mp = \frac{1}{2}\left(1 \mp \gamma_5\right). \tag{A1}$$

The matrix Σ_+ (Σ_-) are the projector onto the quark state with the spin orientated along (against) the chromomagnetic field B_j. The simplest way to prove (27) and (30) consists in the following. Notice, that $[\nabla_\mu(x), \nabla_\nu^*(x)] = 0$. Therefore, if $\phi_0(x)$ is a zero mode of the operator $\hat{\nabla}_\mu(x)$, then for any $y \in R^4$ the field

$$\psi_0(x, y) = \exp\left\{-y_\mu \nabla_\mu^*(x)\right\} \phi_0(x) \tag{A2}$$

is a zero mode either. The zero eigenvalue is strongly degenerate, which is a manifestation of an invariance of the background field under translations and appropriate gauge transformations (see (4)). It is notable that any nonzero eigenvalue of $\hat{\nabla}_\mu(x)$ is also degenerate. Furthermore, using the identity $\sigma_{4j} = -\gamma_5\Sigma_j$, the algebra of γ-matrices and an (anti-)self-duality of the tensor $B_{\mu\nu}$, one can get

$$\sigma_{\mu\nu}B_{\mu\nu} = -4BP_\mp(\Sigma_+ - \Sigma_-), \tag{A3}$$

$$\gamma_\mu B_{\mu\nu}x_\nu \cdot \sigma_{\alpha\beta}B_{\alpha\beta} = 4iB^2\gamma_\mu x_\mu P_\mp. \tag{A4}$$

Here P_- (P_+) relates to a self-dual (anti-self-dual) field. Substituting (A3) to (A4) and comparing the result with (A4), one arrives at the identities

$$i\gamma_\mu B_{\mu\nu}x_\nu P_\mp \Sigma_+ = B\gamma_\mu x_\mu P_\mp \Sigma_+,$$
$$i\gamma_\mu B_{\mu\nu}x_\nu P_\mp \Sigma_- = -B\gamma_\mu x_\mu P_\mp \Sigma_-. \tag{A5}$$

Using these identities and taking into account (A2), one can check that

$$\hat{\nabla}(x)P_\mp\Sigma_\mp \exp\left\{-\frac{1}{4}\sqrt{n^2}B(x-y)^2 + \frac{i}{2}nx_\mu B_{\mu\nu}y_\nu\right\} = 0, \tag{A6}$$

where Σ_- (Σ_+) corresponds to $n > 0$ $(n < 0)$ with n being an element of the diagonal matrix given in (3). Thus, the squire integrable solution for the zero mode of Dirac operator in the homogeneous (anti-)self-dual field has the form of (30).

Equations (25) and (27) can also be obtained by a straightforward calculation of the quark propagator. Substituting $S = \exp(inx_\mu B_{\mu\nu}y_\nu/2)H(x-y)$ to (23), one gets the following representation for the function H

$$H(z) = \left(i\hat{\nabla} + m\right)\left(-\nabla^2 + m^2 - \frac{1}{2}n\sigma_{\alpha\beta}B_{\alpha\beta}\right)^{-1}\delta(z). \tag{A7}$$

Using the identity (A3), we rewrite (A7) in the form

$$H(z) = \left(i\hat{\nabla} + m\right)\left[\frac{P_\pm}{-\nabla^2 + m^2} + \frac{P_\mp\Sigma_+}{-\nabla^2 + m^2 + 2nB}\right.$$
$$\left. + \frac{P_\mp\Sigma_-}{-\nabla^2 + m^2 - 2nB}\right]\delta(z). \tag{A8}$$

The Schwinger proper time method gives:

$$\frac{1}{-\nabla^2 + M}\delta(z) = \frac{B\sqrt{n^2}}{16\pi^2}\int\limits_0^\infty \frac{ds}{\sinh^2(s)}$$
$$\times \exp\left(-Ms/B\sqrt{n^2} - z^2 B\sqrt{n^2}\coth(s)/4\right).$$

Substitution of this formula to (A8) and the subsequent change of variable $s = -\ln t/2$ lead to the quark propagator (35). At last, in order to get the leading terms (25) and (27) of the quark propagator in the massless limit one needs to use (A6) and the identities

$$\vec{\hat{\nabla}}\, P_\mp \Sigma_\mp \exp\left\{-\sqrt{n^2}Bz^2(1+t)/4(1-t)\right\}$$
$$= t\exp\left\{-\sqrt{n^2}Bz^2(1+t)/4(1-t)\right\}\overleftarrow{\hat{\nabla}}\, P_\mp \Sigma_\mp$$
$$\vec{\hat{\nabla}}\, P_\mp \Sigma_\pm \exp\left\{-\sqrt{n^2}Bz^2(1+t)/4(1-t)\right\}$$
$$= t^{-1}\exp\left\{-\sqrt{n^2}Bz^2(1+t)/4(1-t)\right\}\overleftarrow{\hat{\nabla}}\, P_\mp \Sigma_\pm$$

in (35). Here, the upper (lower) sign in $\Sigma_\mp$ and $\Sigma_\pm$ corresponds to $n > 0$ $(n < 0)$.

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